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MODE RECONSTRUCTION IN A ONE-PARAMETER FAMILY OF OPTIMAL CONTROL PROBLEMS†

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A two-dimensional free-time optimal control problem with an integral performance index which depends on a scalar parameter is considered. For zero value of the parameter the problem reduces to the well-known Fuller problem with the chattering phenomenon involving infinitely many switching points in a finite time [1]. A preliminary analysis, based on the maximum principle, shows that such a regime is maintained up to a certain critical value of the parameter, after which one has a two-switching regime involving a first-order singular arc. The optimality of the above-mentioned regimes is proved using dynamic programming. A group-invariant analysis of Bellman's equation, similar to that in [2], reveals the structure of the twice differentiable Bellman's function involving several unknown constants. These are found numerically from an algebraic system of equations. © 2001 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Consider an optimal control problem given by the following differential equations, control parameter constraints, initial and terminal conditions

$$\dot{x} = y,\ \ \dot{y} = u,\ \ 0 \leqslant t \leqslant T,\ \ |u| \leqslant 1 \tag{1.1}$$

$$x(0) = x^0,\ \ y(0) = y^0;\ \ x(T) = 0,\ \ y(T) = 0$$

Here T is the free endpoint of the controlled motion and μ is the scalar control parameter. The following functional, defined on the paths of system (1.1), is considered as the performance index (where L is a real-valued parameter)

$$J = \int_0^T x^2(t)[Lu(t) + 1]dt,\ \ |L| \leqslant 1 \tag{1.2}$$

The set of admissible controls consists of piecewise-continuous functions $u(t)$, $0 \leqslant t \leqslant T$, subject to the constraints in (1.1), i.e. having values in the range $[-1, 1]$. The problem under consideration is to minimize functional (1.2) over the admissible trajectories of system (1.1).

When $L = 0$ we have the so-called Fuller's problem, investigated in [1, 2]. A complete investigation of Fuller's problem and some other optimal control problems involving chattering phenomenon can be found in [3]. Note that for $|L| < 1$ the integrand in functional (1.2) is positive.

The restriction $|L| \leqslant 1$ on the parameter L makes functional (1.2) non-negative definite, while for other values of L, $|L| > 1$, a minimum of (1.2) does not exist and one can find a sequence of admissible controls such that functional (1.2) tends to $-\infty$.

Some computations for this paper were carried out in terms of another parameter a related to L by the equalities

$$a = \frac{1+L}{1-L},\ \ -1 \leqslant L < 1;\ \ L = \frac{a-1}{a+1},\ \ 0 \leqslant a < \infty \tag{1.3}$$

Under this transformation functional (1.2), up to a positive multiplier $a+1$, can be represented in the form

$$J = \frac{1}{2}\int_0^T x^2(t)[(1+u)a + 1 - u]dt \tag{1.4}$$

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Note that it is sufficient to consider only positive values of the parameter L, since when $L < 0$, one can replace it by a positive parameter by changing the sign of the control parameter and the phase coordinates in (1.1).

In terms of a the Hamiltonians of the problem and the corresponding Poisson brackets take a simpler form.

2. CONSTRUCTION OF THE SWITCHING CURVE USING THE MAXIMUM PRINCIPLE

The optimal phase portrait of problem (1.1), (1.2) is generally characterized by a switching curve M that divides the (x, y) plane into two domains N^+ and N^-, in which the optimal control takes the value $u = +1$ and $u = -1$ respectively.

We assume, as in case when $L = 0$, see [1, 2], that the curve M for other values of L also consists of two half-parabolas M^+ and M^-, which, generally speaking, are asymmetric

$$M^+: \ x = K_+y^2, \ y \geqslant 0; \ M^-: \ x = K_-y^2, \ y \leqslant 0 \tag{2.1}$$
$$K_+ < 0, \ \ K_- > 0$$

Hence, the half-parabola M^+ lies in the second quadrant of the (x, y) plane, while M^- lies in the fourth quadrant, Fig. 1.

It is well known that for $L = 0$ we have $C = -K = 0.4446$, [1, 2]. Note that the asymmetric switching curve of the form (2.1) found in [4] is due to asymmetric constraints on the control parameter $a \leqslant u \leqslant b$, whereas in the present paper it is due to the modified functional. Unlike the situation in [4] such a functional also causes a phase portrait bifurcation, as shown below.

One can find the constants K_+ and K_- using the maximum principle [5]. To retain a single system of notation we will change the signs of the adjoint variables p and q and apply the minimum principle. Thus, $p = -\phi$ and $q = -\psi$, where ϕ and ψ are adjoint variables of the maximum principle.

The Hamiltonian and its extremal values have the form

$$H(x, y, p, q, u) = py + qu + x^2(Lu + 1)$$
$$\min_u H = \min[F^+, F^-] = py + x^2 - \left|q + Lx^2\right| \tag{2.2}$$
$$F^\pm = py + x^2 \pm q \pm Lx^2, \ \ u^* = -\operatorname{sign}(q + x^2 L)$$

Here the maximization procedure is replaced by minimization due to the change in the sign of p and q; u^* is the optimal control.

Suppose the optimal trajectory, as in the case when $L = 0$, consists of parts of two families of parabolas corresponding to $u = +1$ and $u = -1$. Consider a part of such a trajectory starting at the initial point $(x_*, y_*) \in M^+$, reaching the curve M^- and lying in the domain N^- where $u = -1$. Thus, the point x_*, y_* and the corresponding quantities p_*, q_* satisfy the conditions

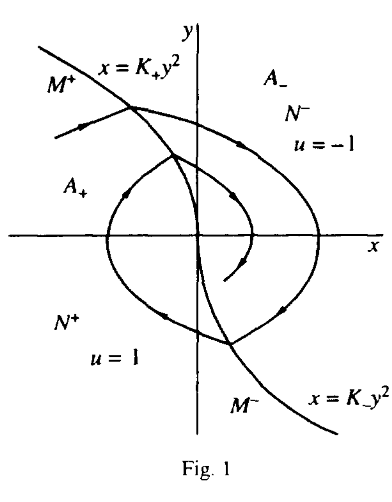

Fig. 1

$$x_* = K_+ y_*^2, \quad q_* + x_*^2 L = 0 \tag{2.3}$$
$$F^{\pm} = p_* y_* + x_*^2 \pm q_* \pm L x_*^2 = 0$$

Here the switching condition

$$q + Lx^2 = 0 \tag{2.4}$$

is used as well as the identity $H(t) = 0$, $0 \leqslant t \leqslant T$, which arises from the maximum principle. Using relations (2.3) we can express the quantities x_*, p_* and q_* in terms of y_* as follows:

$$x_* = K_+ y_*^2, \quad p_* = -K_+^2 y_*^3, \quad q_* = -LK_+^2 y_*^4 \tag{2.5}$$

Integrating the system of equations of the maximum principle [5] for the domain N^-

$$\dot{x} = y, \quad \dot{y} = -1, \quad \dot{p} = -F_x^- = -2x(1-L), \quad \dot{q} = -F_y^- = -p$$

with initial conditions (2.5) at $t = 0$ we obtain expressions for y, x, p and q as polynomials in t of degree 1, 2, 3 and 4 respectively. In particular, $y = -t + y_*$. At the instant of time $t = t_1$ when the trajectory reaches the curve M^- the following two equalities must be satisfied: the switching condition (2.4) and the equation $x = K_- y^2$ of the curve M^-. Using polynomial expressions for x, y and q and introducing the notation

$$\Lambda^{\pm}(\tau) = \frac{4L \mp 1}{12}\tau^3 \mp \frac{4L-1}{3}\tau^2 + (K_\pm \mp 2K_\pm L + L)\tau + 2K_\pm L + K_\pm^2$$

we can represent these two equalities in the following form

$$\tau = 1 + \left(\frac{1+2K_+}{1+2K_-}\right)^{1/2}, \quad \Lambda^+(\tau) = 0; \quad \tau = \frac{t}{y_*} \tag{2.6}$$

Expression (2.6) for $\tau = t/y_*$ corresponds to the maximum root of the quadratic equation $x(t) = K_- y^2(t)$ in t, while the minimum root gives the print of intersection with that part of the parabola $x = K_- y^2$ which is not used in the construction.

Similarly, considering the part of the optimal path coming from M^- to M^+ through the domain N^+, we can obtain another pair of equations

$$\rho = -1 - \left(\frac{1-2K_-}{1-2K_+}\right)^{1/2}, \quad \Lambda^-(\rho) = 0; \quad \rho = \frac{t}{y_*} \tag{2.7}$$

System (2.6), (2.7) of four equations in the four unknowns K_+, K_-, τ and ρ was used to calculate the relations $K_+(L)$ and $K_-(L)$; the latter relation is represented in Fig. 2. In particular, for $L = 0$ we obtain $K_- = -K_+ = 0.4446$, the same as in [1, 2].

System (2.6), (2.7) is only useful for the range of values $|L| < 1/4$. For $|L| \geqslant 1/4$ the chattering regime with infinitely many switchings is replaced by a regime with at most two switchings and with a first-order singular arc.

On a singular segment two equalities are satisfied identically in time, namely, (2.4) and

$$py + x^2 = 0 \tag{2.8}$$

Equality (2.8) follows from switching condition (2.4) and from the vanishing of the Hamiltonian: $H = 0$. Differentiating equality (2.8) with respect to time along the solution of the Hamiltonian system

$$\dot{x} = H_p = y, \quad \dot{y} = H_q = u \tag{2.9}$$
$$\dot{p} = -H_x = -2x(Lu+1), \quad \dot{q} = -H_y = -p$$

we obtain

$$-p + 2xyL = 0 \tag{2.10}$$

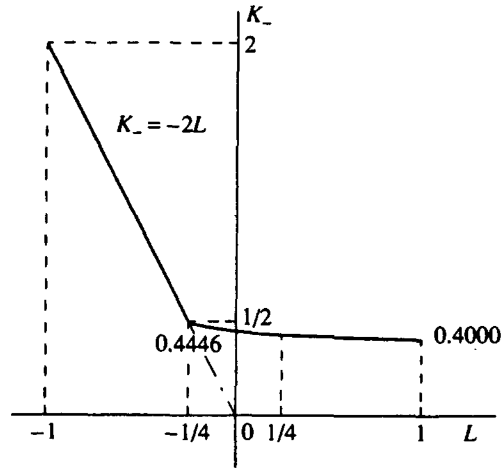

Fig. 2

The latter equation together with (2.8) gives the following equation (of a parabola) for the singular arc

$$x = -2Ly^2 \tag{2.11}$$

Differentiating equality (2.10) along the solutions of system (2.9) and using (2.11) we obtain the equality $x(4Lu + 1) = 0$, from which we get the singular control

$$u_s = -1/(4L) \tag{2.12}$$

Using Hamiltonian (2.2) and the singular control (2.12) it can be shown that the so-called Kelley condition [6] is satisfied on the singular parabola (2.11)

$$\frac{\partial}{\partial u}\frac{\partial^2}{\partial t^2}\frac{\partial H}{\partial u} = -8L^2y^2 \leqslant 0$$

The change in the sign of the inequality here is due to the change in the sign of the adjoint variables. For the invariant form of the Kelley condition see [7].

The singular control (2.12) must satisfy the initial constraint $|u_s| \leqslant 1$, whence we see that the singular regime is only possible when $|L| \geqslant 1/4$. In that case parabola 2.11 takes the role of the branch M^- (the branch M^-) of the switching curve for $L > 1/4$ (for $L < -1/4$).

Hence, for $1/4 \leqslant L \leqslant 1$ we must put $K_+ = -2L$ and determine the coefficient $K_-(L)$ from system (2.7); for $-1 \leqslant L \leqslant -1/4$ we must put $K_- = -2L$ and find $K_+(L)$ from system (2.6).

Computations showed that system (2.6), (2.7) is also compatible for $|L| > 1/4$, but the corresponding values of K_+ and K_- are not elements of the solution of the optimal control problem. This can be verified by dynamic programming.

3. BELLMAN'S EQUATION

Let $V(x, y)$ be the function of the optimal result (Bellman's function) in problem (1.1), (1.2), i.e. the minimum value of functional (1.2) along the trajectories of system (1.1) starting at the point (x, y). The function $V(x, y)$ is defined in the whole (x, y) plane and satisfies the equation

$$\min_u H\left(x, y, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, u\right) = y\frac{\partial V}{\partial x} + x^2 - \left|\frac{\partial V}{\partial y} + Lx^2\right| = 0 \tag{3.1}$$

$$u^* = -\operatorname{sign}\left(\frac{\partial V}{\partial y} + Lx^2\right)$$

In the domains N^- and N^+, into which the switching curve M divides the (x, y) plane, Bellman's function satisfies the equations

$$F^-\left(x,y,\frac{\partial V}{\partial x},\frac{\partial V}{\partial y}\right)\equiv y\frac{\partial V}{\partial x}-\frac{\partial V}{\partial y}-Lx^2+x^2=0 \tag{3.2}$$

$$(x,y)\in N^-,\quad \frac{\partial V}{\partial y}+Lx^2>0,\quad u^*=-1$$

$$F^+\left(x,y,\frac{\partial V}{\partial x},\frac{\partial V}{\partial y}\right)\equiv y\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y}+Lx^2+x^2=0 \tag{3.3}$$

$$(x,y)\in N^+,\quad \frac{\partial V}{\partial y}+Lx^2<0,\quad u^*=+1$$

The Hamiltonians H, F^+ and F^- are defined in (2.2). We will denote the restriction of Bellman's function to the domains N^- and N^+ by $V^-(x,y)$ and $V^+(x,y)$:

$$V^{\pm}(x,y)=V(x,y),\quad (x,y)\in N^{\pm} \tag{3.4}$$

Hence, the function $V^-(x,y)$ (the function $V^+(x,y)$) satisfies Eq. (3.2) (Eq. (3.3)). Equations (3.1)–(3.3) need to have boundary conditions. The terminal condition in (1.1) specifies the following value for Bellman's function at the origin

$$V(0,0)=0 \tag{3.5}$$

This equality will also be considered as the boundary condition for Eqs (3.1)–(3.3). Generally speaking, for the two-dimensional problem regular boundary conditions must be specified on a certain curve rather than at a single point [8]. However, condition (3.5) happens to be sufficient for a unique solution due to a certain degeneracy of the problem.

The solution of problem (3.1), (3.5) will be sought in the class of continuously differentiable functions, i.e. Eqs (3.1)–(3.3) are understood in the classical sense rather than in the generalized one.

A group-invariant analysis enables us to simplify the solution of Bellman's equation. We can verify that the equations and the constraints in (1.1) are invariant under the group of transformations

$$x=\mu^2\bar{x},\quad y=\mu\bar{y},\quad u=\bar{u},\quad t=\mu\bar{t},\quad \mu>0 \tag{3.6}$$

where $\mu>0$ is a scalar parameter. A multiplier μ^5 then arises in functional (1.2) which means that the values of Bellman's function at the corresponding points $V(x,y)$ and $V(\bar{x},\bar{y})$ are connected by the relation

$$V(x,y)=V(\mu^2\bar{x},\mu\bar{y})=\mu^5V(\bar{x},\bar{y}) \tag{3.7}$$

This enables us to express the function of two variables in terms of a function of one variable. Using the parameter $\mu=1/y$ when $y\geqslant 0$ and $\mu=-1/y$ when $y\leqslant 0$ in (3.6) and (3.7), we obtain the representation

$$V(x,y)=y^5\varphi(xy^{-2}),\quad \varphi(z)=\begin{cases} V(z,1), & y>0\\ -V(z,-1), & y<0\end{cases} \tag{3.8}$$

The branches of the function $\varphi(z)$ corresponding to the branches of Bellman's function $V^-(x,y)$ and $V^+(x,y)$, according to relation (3.8), will be denoted by φ^-, φ^+:

$$V^{\pm}(x,y)=y^5\varphi^{\pm}(xy^{-2}),\quad (x,y)\in N^{\pm} \tag{3.9}$$

The function (3.8) satisfies condition (3.5), which is thus a necessary condition for the existence of a self-similar solution of Eq. (3.1). This means that condition (3.5) cannot be used when constructing the functions φ^+ and φ^-.

Equations (3.2) and (3.3) lead to the following ordinary differential equations for the functions φ^+ and φ^-, where $z=x/y^2$,

$$\varphi^{\pm} : \varphi(z)(1 \mp 2z) \pm 5\varphi(z) + z^2(1 \pm L) = 0 \tag{3.10}$$

The general solutions of the linear inhomogeneous equations (3.10) have the form

$$\varphi^{\pm}(z) = A_{\pm}\left|z \mp \frac{1}{2}\right|^{5/2} + 2(1 \pm L)\left(\mp\frac{1}{15} + \frac{1}{3}z \mp \frac{1}{2}z^2\right) \tag{3.11}$$

where $A_{\pm}$ are integration constants. It is assumed here that the point $z = 1/2$ (the point $z = -1/2$) does not lie in the region in which the solution $\varphi^+(z)$ ($\varphi^-(z)$) is defined. These points are singular points of Eqs (3.10) since the coefficients of higher derivatives vanish here. Of these points are internal points of the regions in which the solutions are defined, then the general solutions are two-parameter sets. For example, one can use different values of A_+ for different sides of the point $z = 1/2$, say, A_+ and A_+^*. Note that the two branches corresponding to A_+ and A_+^* match smoothly at $z = 1/2$, giving one single continuously differentiable function. Such a situation is encountered below when finding the constants $A_{\pm}$ and $K_{\pm}$.

Using (3.9) and (3.11) we obtain the following expressions for the values V^+ and V^- of the function V in the domains N^+ and N^-

$$V(x,y) = V^{\pm}(x,y) = A_{\pm}\left|\frac{1}{2}y^2 \mp x\right|^{5/2} + 2(1 \pm L)\left(\mp\frac{1}{15}y^5 + \frac{1}{3}xy^3 \mp \frac{1}{2}x^2y\right) \tag{3.12}$$

$$(x,y) \in N^{\pm}$$

Here, generally speaking, each of the regions $y \geqslant 0$ and $y \leqslant 0$ must have its own constant A_+ or A_-. However, the requirement that the functions $V^+(x, y)$ and $V^-(x, y)$ must be continuous at $y = 0$ leads to a common value for these constants, which is also assumed in (3.12).

Hence, the construction of the smooth function $V(x, y)$ reduces to finding the constants $A_{\pm}$ and $K_{\pm}$. The existence of such a smooth Bellman function, as follows, for example, from the results of [9], proves the optimality of the synthesis described in Sections 2 and 3.

4. DETERMINATION OF THE PARAMETERS $A_{\pm}$ AND $K_{\pm}$

When calculating $A_{\pm}$ and $K_{\pm}$ the parameter L is considered to be given. The amount of computation can be reduced thanks to the following symmetry when the sign of L is changed

$$L \to -L: \ A_{\pm} \to A_{\mp}, \ \ K_{\pm} \to -K_{\mp}, \ \ V(x,y) \to V(-x,-y) \tag{4.1}$$

In particular, it is sufficient to find the relation $K_-(L)$, $|L| \leqslant 1$, in order to obtain $K_+(L) = -K_-(-L)$.

To compute $A_{\pm}$ and $K_{\pm}$ in the range of values $|L| \leqslant 1/4$ one has to consider the following system of four equations

$$V^+(x,y) = V^-(x,y), \quad \frac{\partial V^+}{\partial y} + Lx^2 = 0 \quad (x = K_-y^2) \tag{4.2}$$

$$V^+(x,y) = V^-(x,y), \quad \frac{\partial V^-}{\partial y} + Lx^2 = 0 \quad (x = K_+y^2) \tag{4.3}$$

As can be seen from these values of x, Eq. (4.2) is considered along the curve M^-, while Eq. (4.3) is considered along the curve M^+. On substituting $x = Cy^2$ and $x = Ky^2$ into (4.2) and (4.3), a common factor y^4 or y^5 occurs, after cancelling which the following four equations in terms of the parameters $A_{\pm}$, $K_{\pm}$ and L remain

$$\mp A_+\left|\frac{1}{2} - K_{\mp}\right|^{5/2} + 2(1 + L)\left(-\frac{1}{15} + \frac{1}{3}K_{\mp} - \frac{1}{2}K_{\mp}^2\right) =$$

$$= \mp A_-\left|\frac{1}{2} + K_{\mp}\right|^{5/2} + 2(1 - L)\left(\frac{1}{15} + \frac{1}{3}K_{\mp} + \frac{1}{2}K_{\mp}^2\right) \tag{4.4}$$

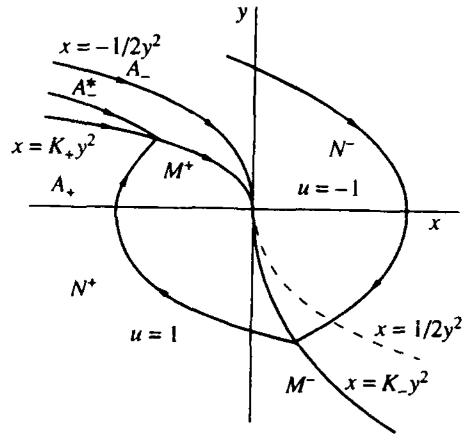

Fig. 3

$$\mp\frac{5}{2}A_{\pm}\operatorname{sign}\left(K_{\mp}\mp\frac{1}{2}\right)\left|K_{\mp}\mp\frac{1}{2}\right|^{3/2}+2(1\pm L)\left(\mp\frac{1}{3}+K_{\mp}\mp\frac{1}{2}K_{\mp}^{2}\right)+LK_{\mp}^{2}=0 \tag{4.5}$$

These equalities represent a system of four transcendental equations in the unknowns $A_{\pm}$ and $K_{\pm}$, that was solved numerically using the MAPLE software package.

For the range of values $L \geqslant 1/4$, by (2.11) we have $K = -2L$. Substituting this value into Eq. (4.7) we can obtain the following value of the constant $A_- = A_-^*$

$$A_-^* = \frac{8\sqrt{2}}{15}\frac{3L-1}{\sqrt{4L-1}} \tag{4.6}$$

The function $V^*(x,y)$, equal to $V^-(x,y)$ in (3.12) with the constant (4.6), represents Bellman's function in the subdomain of N^- which lies between the parabolas M^+ and $x = -y^2/2$ (Fig. 3). In the remaining part of the domain N^- the function $V^-(x,y)$ represents the constant A_- which is found, together with A_+ and K_-, from the system of three equations including (4.4) and (4.5) with upper sign and the modified equation (4.4) with lower sign in which we substitute $A_- = A_-^*$ and $K_+ = -2L$.

Note that when $L = 1/4$ we have $K_+ = -1/2$, and to find $A_{\pm}$ and K_- one has to use a system consisting of both Eqs (4.4) and Eq. (4.5) with the upper sign. The need for the constant A_-^* from (4.6) in Eq. (4.4) disappears (the curve M^+ coincides with $x = -y^2/2$, Fig. 3). Although also $A_-^* \to \infty$ as $L \to 1/4 + 0$, the value of $V^*(x,y)$ tends to a finite limit for $(x,y) \in M^+$: $V^* = (3/80)y^5$.

The values $L \leqslant -1/4$ are analysed similarly using substitution (4.1).

The results of calculations are illustrated in Fig. 2 and by the following numerical data

L	0.0	0.2	0.25	0.4	0.6	0.8	1.0
A_+	0.7640	0.8998	0.9333	1.0291	1.1453	1.2513	1.3492
A_-	0.7640	0.6269	0.5926	0.4894	0.3515	0.2133	0.0749
A_-^*	– –	– –	– –	0.1947	0.5100	0.7119	0.8709
K_+	−0.4446	−0.4801	−0.5000	−0.8000	−1.2000	−1.6000	−2.0000
K_-	0.4446	0.4277	0.4248	0.4174	0.4100	0.4044	0.4000

The graph of the function $K_-(L)$, shown in Fig. 2, has a corner point at $L = -1/4$ when the phase portrait bifurcates. Correspondingly, the graph of $K_+(L)$ has a corner point at $L = 1/4$. Values of $K_-(L)$ for some points in the range $-1/4 \leqslant L \leqslant 1$ were found numerically; in the range $-1 \leqslant L \leqslant -1/4$ the equality $K_-(L) = -2L$ is satisfied.

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